

Testing order restrictions in contingency tables

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Abstract

Several interesting models for contingency tables are defined by a system of equality and inequality constraints on a suitable set of marginal log-linear parameters. After reviewing the most common difficulties which are intrinsic to order restricted testing problems, we propose two new families of testing procedures, based on similar attempts appeared in the econometric literature, in order to increase the probability of detecting several relevant violations of the supposed order relations. One set of procedures is based on the decomposition of the log-likelihood ratio when testing the given set of inequalities and the nested model derived by forcing inequalities into strict equalities. The other set uses the asymptotic joint normal distribution of the estimates of the marginal log-linear parameters to be constrained.

Keywords: Stochastic orderings, chi-bar squared distribution, positive association

1. Introduction

Models for contingency tables involving order restrictions may arise in several contexts. For two-way tables, when both variables have ordered categories, Dardanoni and Forcina (1998) extended the approach of Dykstra et al. (1995) for testing the set of inequalities implied by the assumption that the conditional distributions by row satisfy suitable stochastic orderings and Cazzaro and Colombi (2006) considered testing stochastic ordering by row and columns simultaneously. More generally, inequality constraints arise when we assume that a pair of ordered categorical variables are positively associated (Bartolucci et al., 2007) or that the strength of the association increases with respect to a third variable (Colombi and Forcina, 2001). Order restrictions may also be implied by graphical models containing latent variables when we restrict attention to the marginal distribution of the observed variables, for a general treatment see Evans (2013). In the very special case of item response models, Bartolucci and Forcina (2000) considered testing the order relation known as MTP2 which is implied by the assumptions of conditional independence and monotonicity. For a very extensive review of the literature on the subject see Agresti and Coull (2002).

The problem of testing a model defined by a system of linear inequality constraints imposed on a set of marginal log-linear parameters against the saturated model is a difficult one because, in the most common formulations, the null hypothesis states that the true parameter value lies into a convex cone and may be on the boundary (Shapiro, 1985). The problem of determining a least favorable distribution within the null may be a difficult one (See Silvapulle and Sen, 2005, Sec. 4.8.5) because this does not necessarily coincides with the model where all inequalities hold as strict equalities. However, the more restrictive hypothesis where all inequalities hold as strict equalities may be of interest on its own, for instance, the assumption of positive quadrant dependence in a two-way table (Bartolucci et al., 2001), defined by the constraint that all the log-odds ratios of type global are non negative, corresponds to a convex cone whose vertex is the model of independence. The problem of testing that a system of linear inequalities may be replaced by equalities against the assumption that at least one inequality is strict (see for example Silvapulle and Sen, 2005, Sec. 4.3) is closely related to the problem of testing the corresponding set of inequalities against the saturated model in the sense that the likelihood ratio statistics for testing a system of equalities against the unrestricted model may be partitioned into two

components, one for testing equalities against inequalities and the other for testing inequalities against the saturated model, whose asymptotic distributions are mixture of chi squared random variables.

It is well known that testing a set of equalities against a corresponding set of inequalities, may lead to reject the null even if the inequality constraints are clearly violated in the data (see Agresti and Coull, 2002, 3.3). This fact has been used by Kateri and Agresti (2013) as an argument against using frequentist approaches in this context. The problem has been studied in detail within the econometric literature and valid frequentist solutions have been proposed. For instance, in the context of testing Lorenz curve orderings, Dardanoni and Forcina (1999) designed a procedure based on the joint distribution of the two components of the likelihood ratio statistics mentioned above. Starting from similar concerns, Bennet (2013) proposed a procedure based on a multiple comparisons approach. In this paper we propose an extension of both procedures and compare their merits with a simulation. It turns out that the procedures based on the joint distribution of the likelihood ratio statistics performs much better against most alternatives.

In section two we introduce the notation and present a formal statement of the general problem. In section 3 we review the basic features of the problem of testing a set of inequality constraints and of the chi-bar squared distribution. In section four we present two new sets of testing procedures, one based on the two likelihood ratio statistics for testing equalities against inequalities and inequalities against the saturated model and the other on Multiple Comparisons approaches. Two real data sets are analyzed in Section 4 to illustrate the procedures. The simulation study of section five provides clear evidence that the proposed approaches can provide satisfactory solutions, though the procedures based on the Likelihood ratio statistics perform substantially better.

2. Notation and preliminary results

Consider a contingency table determined by the joint distribution of d discrete random variables and let \mathbf{i} be a vector of d indices; $p_{\mathbf{i}}$ denote the probability that an observation falls in cell \mathbf{i} and \mathbf{p} the vector whose elements $p_{\mathbf{i}}$ are arranged in lexicographic order relative to \mathbf{i} . We assume that the elements of \mathbf{p} are strictly positive. Let $\boldsymbol{\eta}$ be a vector of marginal log-linear parameters as defined in Bartolucci et al. (2007) and we assume that the mapping between $\boldsymbol{\eta}$ and \mathbf{p} is a diffeomorphism. Any such vector may be defined as $\boldsymbol{\eta} = \mathbf{C} \log(\mathbf{M}\mathbf{p})$ where \mathbf{M} is a matrix of 0's and 1's which produce the appropriate marginal or aggregated probabilities and \mathbf{C} is a matrix of row contrasts. This formulation allow to consider ordinary log-linear parameters as well as logits and higher order interactions of type global or continuation. Assume that $\boldsymbol{\eta}$ is contained in an open subset of \mathbb{R}^{t-1} , where t is the number of cells of the table.

Let \mathcal{C} denote a closed convex cone and assume that $\mathcal{L}_0, \mathcal{L}_1$ are, respectively, the linear space of largest and smallest dimensions such that $\mathcal{L}_0 \subset \mathcal{C} \subset \mathcal{L}_1$. Let $\dim(\mathcal{L}_0) = q$, $\dim(\mathcal{L}_1) = r$; finally let $\mathcal{S} \supseteq \mathcal{L}_1$ be the parameter space of the saturated model. Define

$$H_0 : \boldsymbol{\eta} \in \mathcal{L}_0, \quad H_1 : \boldsymbol{\eta} \in \mathcal{C}, \quad H_2 : \boldsymbol{\eta} \in \mathcal{S}$$

and note that H_1 will usually be defined by a set of equality and inequality constraints while H_0 is obtained by turning all inequalities into strict equalities.

3. Hypotheses testing

Let L_{01} be the log-likelihood ratio for testing H_0 against H_1 and L_{12} the log-likelihood ratio for testing H_1 against H_2 . It is well known that the asymptotic distribution of $L_{02} = L_{01} + L_{12}$ is χ_s^2 , where $s = t - q - 1$. Let \mathbf{F}_0 be the expected information matrix of $\boldsymbol{\eta}$ under H_0 and $\mathbf{V}_0 = \mathbf{F}_0^{-1}$. It can also be shown (see for example Silvapulle and Sen, 2005, 4.3) that

$$Pr(L_{01} > c | H_0) = \sum_q^r w_j(\mathbf{V}_0, \mathcal{C}) Pr(\chi_{j-q}^2 > c), \quad (1)$$

where χ_j^2 denotes a chi square random variable with j degrees of freedom. The above distribution, known as chi-bar squared, depends on the probability weights $w_j(\mathbf{V}_0, \mathcal{C})$ whose definition and computation we discuss below. It can be shown that, under H_0 , L_{12} is asymptotically distributed like the sum of a χ_{t-1-r}^2 and a chi-bar squared with weights $w_j(\mathbf{V}_0, \mathcal{C})$ used in the reverse order (see for instance Silvapulle and Sen, 2005, 4.8.5.). Let $\hat{\boldsymbol{\eta}}$ denote the unrestricted maximum likelihood estimate; an interesting geometric interpretation is that L_{01} and L_{12} are asymptotically equivalent to the squared norm of the projection of $\hat{\boldsymbol{\eta}}$ onto, respectively, the convex cone defined by H_1 and onto its dual, in the metric defined by \mathbf{F}_0^{-1} . From this, the following simple expression for the asymptotic joint distribution of L_{01} and L_{12} can be derived (see for example Dardanoni and Forcina, 1999, Lemma 2):

$$Pr(L_{01} \leq c_1, L_{12} \leq c_2 | H_0) = \sum_q^r w_j(\mathbf{V}_0, \mathcal{C}) Pr(\chi_{j-q}^2 \leq c_1) Pr(\chi_{t-j-1}^2 \leq c_2) \quad (2)$$

The previous joint distribution will play a key role in the next section where we consider testing procedures which use the statistics L_{01} and L_{12} simultaneously.

3.1. The probability weights

Fast and accurate computation of the weights $w_j(\mathbf{V}_0, \mathcal{C})$ is crucial to the use of the testing procedures of section 4.1 below, so we discuss two alternative methods that can be used to evaluate these probabilities. For simplicity, we restrict to the context where $r = t - 1$, meaning that H_1 is defined only by inequality constraints. The more general case where equality constraints are also present, so that $\mathcal{L}_1 \subset \mathcal{S}$, can be reduced by replacing \mathbf{y} with its projection onto \mathcal{L}_1 and $t - 1$ with r , see Shapiro (1988) for a detailed treatment.

If we are prepared to assume that the distribution of L_{01} and L_{12} should be determined under H_0 , the probability weights may be defined as follows. Let \mathbf{x} be distributed as multivariate normal $\mathcal{N}(\mathbf{0}, \mathbf{V}_0)$, then $w_j(\mathbf{V}_0, \mathcal{C})$ is the probability that the projection of \mathbf{x} onto \mathcal{C} falls on a face which spans a linear space of dimension j (see for example Silvapulle and Sen, 2005, Prop. 3.6.1). Recent results (Genz and Bretz, 2009) on multivariate normal integrals, make computation of exact weights fast and accurate for moderate values of r . An algorithm for computing the probability weights derived from (Kudo, 1963, pag 409-416) is implemented in the R-packages **ic-infer** (Grömping, 2010) and **hmmm** (Colombi et al., 2014); a slightly different implementation is outlined in the Appendix. However when the number of inequalities is, say, larger than 15, exact estimation becomes very hard. For this reason, Dardanoni and Forcina (1998) suggested a simulation procedure by which a reasonably large number of sample points from the appropriate normal distribution are projected onto \mathcal{C} ; then $w_j(\mathbf{V}_0, \mathcal{C})$ is estimated by the proportion of sample points falling on any face of \mathcal{C} of dimension j .

The following method, which extends a similar one outlined by Dardanoni and Forcina (1998), can be used to determine the minimum number of sample points required for accurate estimation of probability weights. Let \mathbf{w} be the vector with elements $w_j(\mathbf{V}_0, \mathcal{C})$ and $\hat{\mathbf{w}}$ its estimate. Let \mathbf{c} be the vector with elements $P(\chi_j^2 \leq c_1)P(\chi_{q-j}^2 \leq c_2)$, where c_1, c_2 are determined as in the tunable LR testing procedure of Definition 3 below, and let $\mathbf{v} = \mathbf{c}'[\text{diag}(\mathbf{w}) - \mathbf{w}\mathbf{w}']\mathbf{c}$. Suppose we require that the estimation error $\mathbf{c}'(\hat{\mathbf{w}} - \mathbf{w})$ of estimating the probability of accepting H_0 when true must satisfy the condition

$$P(|\mathbf{c}'(\hat{\mathbf{w}} - \mathbf{w})| \leq \epsilon) \leq 1 - 2\delta.$$

By using the multivariate normal approximation to the multinomial and elementary calculations, we have that the number of points to be projected should not be smaller than $v(z_{1-2\delta}/\epsilon)^2$ where $z_{1-2\delta}$ is the $1 - 2\delta$ percentile of the standard normal distribution.

In general (see Silvapulle and Sen, 2005, 4.3.1) nuisance parameters are going to affect \mathbf{V}_0 and thus the probability weights. The formally correct procedure would be to search for the least favorable null distribution, a task which, however, may be very hard. In addition, it often turns out that the values of the nuisance parameters that produce the least favorable distribution are very extreme and substantially different from any plausible estimate obtained from the data. An alternative procedure

would be to compute the null distribution after replacing the nuisance parameters with their maximum likelihood estimate; a simulation study (Dardanoni and Forcina, 1998, 4.5) indicates that the p -value computed in this way is sufficiently close to the one computed at the true value of the nuisance parameters when the sample size is moderately large.

In certain contexts one could remove the dependence on nuisance parameters by conditioning; for instance, if we are interested in the dependence structure of a two way table, we might condition to the row and columns totals, an approach explored by Bartolucci et al. (2001). The resulting null distributions are, however, hard to handle even with the power of modern computers and can be applied for small sample sizes and in specific contexts.

4. Testing procedures

In this section we present two different families of testing procedures which may lead to one of the following decisions: (i) accept H_0 , (ii) reject H_0 in the direction of H_1 , meaning that there is strong evidence to support the assumed inequality constraints, (iii) reject H_0 in the direction of H_2 , if there is convincing evidence that the inequality constraints are violated. As mentioned in the introduction, if we considered the problem of testing H_1 against the saturated model on its own, then we would face the additional problem of having to determine the least favorable distribution within H_1 : Wolak (1991) has shown that, with non linear problems, the least favorable distribution may not coincide with H_0 . In this paper we avoid this complication because the two testing problems (H_0 against H_1 and H_1 against H_2) are combined into a single testing problem, as detailed in the sub-sections below.

4.1. Likelihood ratios

As emphasized, for instance, by Agresti and Coull (2002), the evidence in favour of H_1 provided by L_{01} may be highly misleading because a large value of this statistic, which would lead to reject H_0 in favour of H_1 , is compatible with substantial violations of H_1 . A geometric explanation is that this event will happen whenever $\hat{\eta}$ is far away from \mathcal{L}_0 and is not contained neither in \mathcal{C} , nor into its dual. Thus, before deciding that the assumed set of inequality constraints are satisfied, one should also examine L_{12} : a large value of this statistic provides evidence that H_1 is violated in the direction of H_2 .

Consider first the following procedure:

Definition 1. *The naive procedure:*

1. accept H_0 if $L_{01} \leq c_1$ where $Pr(L_{01} \leq c_1 | H_0) = 1 - \alpha_1 - \alpha_2$,
2. reject H_0 in favour of H_1 if $L_{01} > c_1$ and $L_{12} \leq c_2$, where $Pr(L_{01} > c_1, L_{12} \leq c_2 | H_0) = \alpha_1$,
3. otherwise reject H_0 for H_2 .

This procedure is closely related to a widely adopted standard approach that rejects H_0 towards H_1 for large values of L_{01} and, when H_0 is rejected, uses the statistics L_{12} to test H_1 against H_2 . However, as far we know, the implementations of this approach, considered in the literature, do not use the joint distribution (2) of the two statistics to control the error rate of a false rejection of H_1 .

An alternative approach, also related to a standard use of L_{01} and L_{12} , is:

Definition 2. *The basic procedure: let c_2 be such that $Pr(L_{12} > c_2 | H_0) = \alpha_2$ then*

1. accept H_0 if $L_{12} \leq c_2$ and $L_{01} \leq c_1$, where $Pr(L_{01} \leq c_1, L_{12} \leq c_2 | H_0) = 1 - \alpha_1 - \alpha_2$,
2. reject H_0 towards H_1 if $L_{01} > c_1$ and $L_{12} \leq c_2$,
3. reject H_0 towards H_2 if $L_{12} > c_2$.

Though the above procedure allows a direct control of the error rates towards H_1 and H_2 and does not suffer from the limitations described by Agresti and Coull (2002), its power in detecting H_2 , when it is true, may still be too low. Dardanoni and Forcina (1999), in the context of comparing economic inequality, suggested a procedure where the amount of protection against rejecting H_0 in favour of H_1 when H_2 holds may be tuned to the specific context. The procedure we describe below is a revised version of their procedure.

Definition 3. *The tunable procedure: let $0 \leq \alpha_{12} \leq \alpha_2$ and let c_2 be such that $Pr(L_{12} > c_2 | H_0) = \alpha_2 - \alpha_{12}$, then*

1. *accept H_0 if $L_{12} \leq c_2$ and $L_{01} \leq c_1$, where $Pr(L_{01} \leq c_1, L_{12} \leq c_2 | H_0) = 1 - \alpha_1 - \alpha_2$,*
2. *reject H_0 in favour of H_1 if $L_{01} > c_1$ and $L_{12} \leq c_{12}$, where $Pr(L_{01} > c_1, L_{12} \leq c_{12} | H_0) = \alpha_1$,*
3. *otherwise reject H_0 for H_2 .*

Note that the tunable procedure reduces to the naive procedure when $\alpha_{12} = \alpha_2$ and is equivalent to the basic procedure when $\alpha_{12} = 0$. The tunable procedure allows to fix the error rates towards each alternative and, in addition, by increasing the tuning probability α_{12} it can decrease the probability of rejecting towards H_1 when some of the assumed constraints are violated. However, larger values of α_{12} correspond also to larger values of c_2 , thus, the side effect is an increasing tendency to accept H_0 even when it should be rejected towards H_2 , which is the main drawback of the naive procedure. This fact will be clarified by the simulation study in the last section.

4.2. Multiple Comparison procedures

There is a collection of procedures which, due to their computational simplicity, have received much attention in Econometric applications about inequality constrained inference problems, see for instance Bishop et al. (1991). To keep notation simple, we restrict attention to the case where H_1 is defined by $\mathbf{D}\boldsymbol{\eta} \geq \mathbf{0}$, where the $k \times (t-1)$ matrix \mathbf{D} is of full row rank. Let $\boldsymbol{\tau} = \mathbf{D}\boldsymbol{\eta}$ and $\hat{\boldsymbol{\tau}}$ denote the unrestricted maximum likelihood estimator. Let n denote the sample size; under H_0 and the usual regularity conditions, it follows that the asymptotic distribution of $\mathbf{z} = \sqrt{n}\hat{\boldsymbol{\tau}}$ is multivariate normal with covariance matrix $\boldsymbol{\Sigma}_0 = \mathbf{D}\mathbf{V}_0\mathbf{D}'$. Let $\max(\mathbf{z})$ and $\min(\mathbf{z})$ denote the greatest and the lowest component respectively of \mathbf{z} . The following approaches have in common the fact that the different rejection regions are based on the $\max(\mathbf{z})$ and $\min(\mathbf{z})$ statistics.

Definition 4. *The naive Multiple Comparisons procedure: let c be such that $Pr(\min(\mathbf{z}) \geq -c, \max(\mathbf{z}) \leq c | H_0) = 1 - \alpha$, then*

1. *accept H_0 if $\max(\mathbf{z}) \leq c$ and $\min(\mathbf{z}) \geq -c$*
2. *reject H_0 for H_1 if $\max(\mathbf{z}) > c$ and $\min(\mathbf{z}) \geq -c$,*
3. *reject H_0 for H_2 otherwise.*

As noted by Dardanoni and Forcina (1999), this procedure has the limitation that the probabilities of rejecting H_0 towards H_1 and H_2 cannot be controlled. Bennet (2013), in the context of comparing income inequality, has recently proposed a procedure which tries to overcome this limitation. Though, formally, Bennet's procedure is based on the one-sided Kolmogorov-Smirnov statistic, it can be adapted to the present context and formulated in terms of the asymptotic normal distribution of the vector \mathbf{z} , defined above.

Definition 5. *Bennet's procedure: let c_1 be such that $Pr(\min(\mathbf{z}) \geq -c_1, \max(\mathbf{z}) \leq c_1 | H_0) = 1 - \alpha_1 - \alpha_2$, then*

1. *accept H_0 if $\max(\mathbf{z}) \leq c_1$ and $\min(\mathbf{z}) \geq -c_1$,*
2. *reject H_0 towards H_1 if $\max(\mathbf{z}) > c_1$ and $\min(\mathbf{z}) > -c_2$ where $Pr(\max(\mathbf{z}) > c_1, \min(\mathbf{z}) > -c_2 | H_0) = \alpha_1$,*
3. *reject H_0 towards H_2 otherwise.*

In this procedure $\alpha = \alpha_1 + \alpha_2$ is the total probability of rejecting H_0 when true; Bennet sets $\alpha_1 = \alpha\beta$ and $\alpha_2 = \alpha(1 - \beta)$, with $\beta = \frac{\alpha_1}{\alpha}$.

We now present a more flexible procedure which, for any preassigned value of α_1, α_2 , like in Definition 3 above, allows to increase the level of protection against stating that H_1 holds when one or more inequalities are violated in the population.

Definition 6. *The tunable Multiple Comparisons (MC) procedure: let $0 \leq \alpha_{12} \leq \alpha_2$ and c_2 be such that $Pr(\min(\mathbf{z}) < -c_2 | H_0) = \alpha_2 - \alpha_{12}$ and then*

1. *accept H_0 if $\min(\mathbf{z}) \geq -c_2$ and $\max(\mathbf{z}) \leq c_1$, where $Pr(\max(\mathbf{z}) \leq c_1, \min(\mathbf{z}) \geq -c_2 | H_0) = 1 - \alpha_1 - \alpha_2$,*
2. *reject H_0 in favour of H_1 if $\max(\mathbf{z}) > c_1$ and $\min(\mathbf{z}) \geq -c_{12}$, where $Pr(\max(\mathbf{z}) > c_1, \min(\mathbf{z}) \geq -c_{12} | H_0) = \alpha_1$,*
3. *otherwise reject H_0 for H_2 .*

A completely different approach to testing inequality constraints, which provides full protection against stating that the assumed inequality constraints hold when they are violated, based on $\min(\mathbf{z})$, was proposed by Sasabuchi (1980) and has been considered by Kateri and Agresti (2013). This procedure has, however, very serious drawbacks (see Dardanoni and Forcina, 1998, 4.6).

Multiple comparison procedures are computationally simpler for two reasons: (i) they can be applied without the need to fit the inequality constrained model and (ii) they do not require computation of the probability weights of the chi bar squared distribution and exploit modern advances in the computation of probability integrals for the multivariate normal distribution. Critical values for the previous procedures rely on the computation of the probability $\Phi(\mathbf{a}, \mathbf{b}, \Sigma_0)$ that a multivariate normal $N(\mathbf{0}, \Sigma_0)$ lies in the rectangle $[\mathbf{a}, \mathbf{b}]$. For a survey of this problem in the context of multiple comparison procedures see Bretz et al. (2011) chapter 3 and Genz and Bretz (2009) chapter 6. Because α_1 , α_2 and α_{12} are probabilities defined under the null hypothesis, in order to compute the critical values, the unknown Σ_0 must be replaced by its estimate $\hat{\Sigma}_0$ under H_0 . Alternatively the elements of \mathbf{z} can be divided by the standard error estimated under H_0 and the critical values computed by evaluating the previous multi-normal integrals using the correlation matrix corresponding to $\hat{\Sigma}_0$.

5. Examples

Table 1 below was used by Kateri and Agresti (2013) as an instance of a context where the statistic L_{01} can give misleading evidence in favour of H_1 when we test H_0 (independence) against H_1 (all local log-odds ratios are non negative). Here we have $L_{01} = 7.89$ and $L_{12} = 1.75$, so, when we compare these

Table 1: Treatment and Extent of trauma due to subarachnoid hemorrhage

	Death	Veget	Major	Minor	Recov
Placebo	59	25	46	48	32
Treated	135	39	147	169	102

statistics with the critical values of several tunable LR procedures computed with exact weights and displayed in table 5 for $\alpha_1 = 0.02$ and $\alpha_2 = 0.03$, we see that all procedures reject H_0 in favour of H_2 , except when the tuning parameter is $\alpha_{12} = 0$. Thus, in this case, a procedure with $\alpha_{12} = 0.015$ seems to be a reasonable choice. For comparison, we also apply a set of multiple comparisons procedures. The minimum and maximum unconstrained estimates of the log-odd ratios (studentized using the

Table 2: Critical values for the data in Table 1

α_{12}	0	0.015	0.02	0.025	0.028	0.03
LR tunable procedures						
c_2	8.87	10.44	11.34	12.88	14.89	Inf
c_1	6.83	5.70	5.43	5.19	5.07	4.98
c_{12}	8.87	1.46	1.16	0.95	0.86	0.81
MC tunable procedures						
c_2	2.43	2.67	2.81	3.03	3.29	Inf
c_1	2.48	2.32	2.29	2.26	2.25	2.24
c_{12}	2.43	1.84	1.74	1.67	1.64	1.62

standard errors estimated under H_0) are equal to -1.168 and 2.186 respectively. These statistics are

assessed against the critical values of several tunable MC procedures displayed in Table 5. All the procedures accept H_0 , a result which shows that the MC procedures tend to be more conservative.

As a second example, we analyze the data in Table 3 from Bartolucci et al. (2001); this is a $2 \times 5 \times 5$ contingency table where a sample of 2904 males were classified according to two age classes and 5 ordered categories of their own occupational prestige (OP) and that of their fathers. The first issue is whether association between father's and son's OP, measured by log-odds ratio of a suitable type, is stronger when sons are older; so, this is equivalent to assume that log-odds ratios increase with son age. If we use the global log-odds ratios, we have $L_{01} = 29.02$ and $L_{12} = 0.29$ and the conclusion is that H_0 is rejected in favour of H_1 by any LR tunable procedure with $\alpha_1 = 0.02$ and $\alpha_2 = 0.03$. For instance, using again exact weights, with $\alpha_{12} = 0$ the critical values are $c_1 = 25.197$, $c_2 = c_{12} = 11.120$, while with $\alpha_{12} = 0.029$ they are $c_1 = 22.117$, $c_2 = 19.731$, $c_{12} = 1.394$; in both cases c_{12} is much larger than the observed value of L_{12} . Instead, if we measure the strength of association by the local log-odds ratios we have $L_{01} = 14.70$ and $L_{12} = 12.55$ and, though we reject again H_0 towards H_1 with $\alpha_{12} = 0$ because $c_1 = 12.214$, $c_2 = c_{12} = 23.820$, we reject towards H_2 if we set $\alpha_{12} = 0.015$ because $c_1 = 10.746$, $c_2 = 26.161$, $c_{12} = 10.785$. This example again shows that the LR tunable procedures allow us to submit H_1 to a severe scrutiny, while the standard procedure would often be too liberal relative to accepting the assumed ordering.

For the same data, it might be of interest to consider also the assumption that, when sons are younger, their OP is stochastically smaller than that of their fathers while the situation reverses when they are older; the idea behind is that, due to welfare improvement, sons will, on the whole, be better off than their fathers, but also that OP improves with age. This assumption compares the marginal distributions of fathers and sons conditionally on the age group of the sons, using logits of type global. Here $L_{01} = 130.00$ and $L_{12} = 5.12$ and, with $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\alpha_{12} = 0$, the critical values are $c_2 = c_{12} = 9.606$ and $c_1 = 14.211$, so H_0 is rejected in favour of H_1 though there are substantial violations in the observed data. Instead, if we set $\alpha_{12} = 0.015$, with the critical values $c_1 = 12.719$, $c_2 = 11.292$, $c_{12} = 1.542$ H_0 must be rejected in favour of H_2 . This example shows that, for large values of L_{01} and small values of L_{12} , the naive LR procedure can give false evidence in favour of H_1 when H_2 is true and that this drawback is eliminated by the introduction of the tuning probability α_{12} .

6. Simulation study

To evaluate the performance of the various procedures, we used a targeted set of simulations concerning the problem of testing independence against the assumption that all the log-odds ratios of type local-local in 3×3 and 3×4 contingency tables are non negative. The sample size n and the number of replications N were fixed to 10,000. To keep the context simple, in all the simulations we set $\alpha_1 = 0.02$ and $\alpha_2 = 0.03$ combined with a range of values for α_{12} . Initial simulations were used to check that all procedures achieved, with high accuracy, the correct size under H_0 , the hypothesis of independence; then, we considered various versions of H_1 and H_2 by selecting specific sets of local odds ratios and constructed the corresponding bivariate distributions having uniform marginals.

The results are displayed in Tables 3-5 below. In all the tables, each column corresponds to a different testing procedures. Within the likelihood ratio (LR) approach, each procedure is identified by the value of the tuning parameter α_{12} , while, within the multiple comparison (MC) approach, results for the Bennet procedure are displayed in addition; for the values of α_1 , α_2 chosen in the simulations, the rejection regions of the naive MC procedure are very close to those of the Bennet procedure, thus results are omitted because they either coincide or the difference is smaller than 10^{-3} . Tables are divided into sub-tables, each corresponding to different experiments, identified by the set of log-odds ratios used to generate the data. Each sub-table has three rows giving the relative frequencies of: (i) acceptance of H_0 , (ii) rejections in favour of H_1 and (iii) rejections in favour of H_2 , for every procedure.

6.1. Power under H_1

All LR procedures seem to have high power even for moderate violations of H_0 in the direction of H_1 , like in Tables 3(a), 4(a); with larger violations power gets very close to 1 as in Tables 3(b), 4(b). Though we explored only the case where all the log-odds ratios were equal, it seems reasonable to expect that the performance is determined by the smallest positive value. The power decreases slightly when α_{12} increases, this is consistent with the fact that larger values of α_{12} provide more protection against rejecting towards H_1 when H_2 is true. The power of MC procedures is considerably smaller with a relatively large error rate in the direction of H_0 .

6.2. Power under H_2

Here the situation is much more complex because violations of H_1 can arise in many different directions and it is unlikely that a procedure can perform best under all possible H_2 alternatives. In the simulations we explored a limited range of possibilities which, however, seem to suggest some general conclusions.

The first result which emerges clearly is that the naive LR procedure cannot be recommended because there are relevant versions of H_2 under which this procedure accepts H_0 with probability close to 1. This happens, for instance, when all log-odds ratios are negative, which means that all inequalities are violated, like in Table 3(g,h). The naive procedure has also a rather poor performance when the negative log-odds ratios dominate in number or in absolute value, like in Tables 3(i), 4(c). In all such cases the naive procedure tends to be terribly conservative. It is true that under few violations of H_1 , like in Tables 3(c,d) and 4(d,e) the naive procedure is the best; however, in these cases, most of the other tuned procedures have high power and the improvement produced by the naive procedure is rather modest and certainly cannot compensate the very bad performance described above. The performance of the tuned LR procedures seem to follow this general pattern: when violations of H_1 in the direction of H_2 are few or the negative log-odds ratios are sufficiently small in absolute value relatively to the positive ones, the procedure with $\alpha_{12} = 0$ tends to reject in the direction of H_1 with a relatively large error rate which, however, can be reduced dramatically by increasing α_{12} , see for instance Tables 3(c,d,j) and 4(d,e,f,g). Instead, when there is some kind of balance between negative and positive log-odds ratios, like in Tables 3(e,k) and 4(i), the LR procedures have a relatively large error rate in the direction of H_0 which increases with α_{12} .

On the whole, MC procedures perform substantially worst than the corresponding LR procedure, with a few exceptions, typically when the negative and positive log-odds ratios are in some kind of balance: compare, for instance, the corresponding entries in Table 5(e,k) and Table 3(e,k). MC procedures also do better when the negative log-odds ratios are small or few in number but only for $\alpha_{12} = 0$; however, already at $\alpha_{12} = 0.015$ the LR procedure performs much better, like in Tables 3, 5(c,d). Usually, performance of MC procedures improves with α_{12} , though they remain too much inferior relative to the corresponding LR procedures. The MC procedures are again substantially inferior when all log-odds ratios are negative, like in Tables 5(g,h). On the whole, Bennet's procedure performs like an MC procedure with α_{12} around 0.010; this, however, may depend on the specific values of α_1 , α_2 used in the simulation. Note also that the naive MC procedure does not do as badly as in the LR context.

Table 3: Likelihood ratio procedures in 3×3 tables

	α_{12}					
	0.000	0.015	0.020	0.025	0.028	0.030
(a), H_1 : 0.08, 0.08, 0.08, 0.08						
H0	0.002	0.001	0.001	0.000	0.000	0.000
H1	0.998	0.975	0.965	0.952	0.945	0.941
H2	0.000	0.025	0.035	0.048	0.055	0.059
(b), H_1 : 0.15, 0.15, 0.15, 0.15						
H0	0.000	0.000	0.000	0.000	0.000	0.000
H1	1.000	1.000	0.999	0.999	0.999	0.999
H2	0.000	0.000	0.001	0.001	0.001	0.001
(c), H_2 : 0.08, 0.08, -0.08, 0.08						
H0	0.172	0.121	0.111	0.104	0.100	0.096
H1	0.769	0.351	0.305	0.271	0.253	0.245
H2	0.059	0.528	0.584	0.626	0.647	0.658
(d), H_2 : 0.15, 0.15, -0.15, 0.15						
H0	0.000	0.000	0.000	0.000	0.000	0.000
H1	0.690	0.092	0.072	0.060	0.054	0.051
H2	0.310	0.908	0.928	0.940	0.946	0.949
(e), H_2 : -0.08, 0.08, 0.08, -0.08						
H0	0.785	0.828	0.845	0.867	0.889	0.919
H1	0.034	0.010	0.009	0.009	0.008	0.008
H2	0.181	0.162	0.146	0.124	0.103	0.074
(f), H_2 : -0.15, 0.15, 0.15, -0.15						
H0	0.305	0.380	0.424	0.495	0.581	0.833
H1	0.031	0.002	0.001	0.001	0.001	0.001
H2	0.664	0.619	0.575	0.503	0.419	0.167
(g), H_2 : -0.08, -0.08, -0.08, -0.08						
H0	0.003	0.007	0.009	0.016	0.033	1.000
H1	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.997	0.993	0.991	0.984	0.967	0.000
(h), H_2 : -0.15, -0.15, -0.15, -0.15						
H0	0.000	0.000	0.000	0.000	0.000	1.000
H1	0.000	0.000	0.000	0.000	0.000	0.000
H2	1.000	1.000	1.000	1.000	1.000	0.000
(i), H_2 : -0.15, 0.04, 0.04, -0.15						
H0	0.047	0.080	0.105	0.154	0.230	1.000
H1	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.953	0.920	0.895	0.846	0.770	0.000
(j), H_2 : -0.04, 0.15, 0.15, -0.04						
H0	0.044	0.028	0.024	0.022	0.020	0.020
H1	0.925	0.468	0.412	0.373	0.353	0.340
H2	0.031	0.504	0.563	0.605	0.627	0.641
(k), H_2 : -0.12, 0.08, 0.08, -0.12						
0	0.469	0.574	0.626	0.708	0.795	0.993
H1	0.001	0.000	0.000	0.000	0.000	0.000
H2	0.530	0.426	0.374	0.292	0.205	0.007

Table 4: Likelihood ratio procedures in 3×4 tables

	α_{12}					
	0.000	0.015	0.020	0.025	0.028	0.030
(a), H_2 : 0.06,0.06,0.06,0.06,0.06,0.06						
H0	0.001	0.000	0.000	0.000	0.000	0.000
H1	0.999	0.960	0.946	0.930	0.921	0.916
H2	0.000	0.040	0.054	0.069	0.078	0.084
(b), H_2 : 0.12,0.12,0.12,0.12,0.12,0.12						
0	0.000	0.000	0.000	0.000	0.000	0.000
H1	1.000	0.998	0.997	0.995	0.994	0.993
H2	0.000	0.002	0.003	0.005	0.006	0.007
(c), H_2 : -0.17,0.15,0.15,-0.17,-0.17,0.15						
H0	0.172	0.248	0.292	0.371	0.474	0.942
H1	0.005	0.000	0.000	0.000	0.000	0.000
H2	0.823	0.751	0.708	0.629	0.526	0.058
(d), H_2 : 0.12,0.12,-0.12,0.12,0.12,0.12						
H0	0.000	0.000	0.000	0.000	0.000	0.000
H1	0.952	0.483	0.428	0.390	0.367	0.356
H2	0.048	0.517	0.572	0.610	0.633	0.644
(e), H_2 : 0.15,0.15,-0.15,0.15,0.15,0.15						
H0	0.000	0.000	0.000	0.000	0.000	0.000
H1	0.902	0.331	0.285	0.251	0.235	0.227
H2	0.098	0.669	0.715	0.749	0.765	0.773
(f), H_2 : -0.6,0.15,0.15,-0.6,-0.6,0.15						
H0	0.021	0.012	0.010	0.009	0.008	0.008
H1	0.932	0.433	0.377	0.337	0.316	0.306
H2	0.047	0.555	0.613	0.654	0.676	0.685
(g), H_2 : 0.15,0.15,-0.15,0.15,0.15,0.15						
H0	0.000	0.000	0.000	0.000	0.000	0.000
H1	0.902	0.331	0.285	0.251	0.235	0.227
H2	0.098	0.669	0.715	0.749	0.765	0.773
(i), H_2 : -0.15,0.15,0.15,-0.15,-0.15,0.15						
H0	0.316	0.386	0.432	0.503	0.579	0.814
H1	0.037	0.003	0.002	0.002	0.001	0.001
H2	0.647	0.611	0.566	0.496	0.420	0.184

Table 5: Multiple comparisons procedures in 3×3 tables

		α_{12}					
	Bennet	0.000	0.015	0.020	0.025	0.028	0.030
(a), H_1 : 0.08, 0.08, 0.08, 0.08							
H0	0.554	0.573	0.464	0.438	0.417	0.406	0.401
H1	0.445	0.427	0.533	0.558	0.578	0.589	0.593
H2	0.001	0.001	0.004	0.004	0.005	0.005	0.006
(b), H_1 : 0.15, 0.15, 0.15, 0.15							
H0	0.008	0.010	0.003	0.002	0.001	0.001	0.001
H1	0.992	0.990	0.997	0.999	0.999	0.999	0.999
H2	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(c), H_2 : 0.08, 0.08, -0.08, 0.08							
H0	0.598	0.608	0.547	0.533	0.520	0.516	0.514
H1	0.249	0.255	0.234	0.230	0.228	0.228	0.229
H2	0.154	0.137	0.219	0.238	0.251	0.256	0.257
(d), H_2 : 0.15, 0.15, -0.15, 0.15							
H0	0.044	0.047	0.025	0.021	0.019	0.018	0.017
H1	0.373	0.424	0.229	0.201	0.185	0.177	0.172
H2	0.583	0.529	0.746	0.778	0.797	0.805	0.810
(e), H_2 : -0.08, 0.08, 0.08, -0.08							
H0	0.659	0.658	0.664	0.670	0.676	0.682	0.685
H1	0.082	0.097	0.048	0.041	0.037	0.035	0.034
H2	0.259	0.245	0.287	0.289	0.287	0.283	0.281
(f), H_2 : -0.15, 0.15, 0.15, -0.15							
H0	0.165	0.164	0.168	0.174	0.182	0.189	0.195
H1	0.071	0.100	0.022	0.016	0.012	0.011	0.010
H2	0.764	0.737	0.810	0.810	0.806	0.800	0.794
(g), H_2 : -0.08, -0.08, -0.08, -0.08							
H0	0.541	0.524	0.675	0.742	0.827	0.902	0.999
H1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.459	0.476	0.325	0.258	0.173	0.098	0.001
(h), H_2 : -0.15, -0.15, -0.15, -0.15							
H0	0.009	0.008	0.038	0.067	0.153	0.313	1.000
H1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.991	0.992	0.962	0.933	0.847	0.687	0.000
(i), H_2 : -0.15, 0.04, 0.04, -0.15							
H0	0.274	0.262	0.360	0.416	0.515	0.621	0.891
H1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.726	0.737	0.640	0.584	0.485	0.379	0.109
(j), H_2 : -0.04, 0.15, 0.15, -0.04							
H0	0.265	0.277	0.219	0.206	0.195	0.189	0.186
H1	0.632	0.649	0.559	0.536	0.520	0.514	0.510
H2	0.104	0.074	0.222	0.258	0.285	0.297	0.304
(k), H_2 : -0.12, 0.08, 0.08, -0.12							
H0	0.460	0.453	0.520	0.554	0.600	0.641	0.695
H1	0.020	0.028	0.006	0.005	0.005	0.004	0.004
H2	0.520	0.520	0.473	0.441	0.396	0.355	0.302

Appendix

Computation of probability weights

In order to compute the weights $w_i(\mathcal{C}, \mathbf{V})$, it may be useful to summarize the geometry of the projection of a random vector $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, \mathbf{V})$ onto a convex cone $\mathcal{C} = \{\boldsymbol{\eta} : \mathbf{D}\boldsymbol{\eta} \geq \mathbf{0}\}$, where \mathbf{D} is a $k \times (t-1)$ matrix of rank k . If \mathbf{H} is the left component of the Cholesky decomposition of the positive definite matrix $\boldsymbol{\Omega} = \mathbf{D}\mathbf{V}\mathbf{D}'$ then $\mathbf{z} = \mathbf{H}^{-1}\mathbf{D}\mathbf{y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_k)$, the transformation $\boldsymbol{\lambda} = \mathbf{H}^{-1}\mathbf{D}\boldsymbol{\eta}$ defines the cone $\mathcal{C}^* = \{\boldsymbol{\lambda} : \mathbf{H}\boldsymbol{\lambda} \geq \mathbf{0}\}$, $\mathcal{C} \in \mathbb{R}^k$, and it holds that: $\min_{\mathbf{D}\boldsymbol{\eta} \geq \mathbf{0}} (\mathbf{y} - \boldsymbol{\eta})' \mathbf{V}^{-1} (\mathbf{y} - \boldsymbol{\eta}) = \min_{\mathbf{H}\boldsymbol{\lambda} \geq \mathbf{0}} (\mathbf{z} - \boldsymbol{\lambda})' (\mathbf{z} - \boldsymbol{\lambda})$. The cone \mathcal{C}^* may also be defined by its generating vectors which are the columns of $\mathbf{U} = \mathbf{H}^{-1}$: a vector \mathbf{z} belongs to \mathcal{C}^* if $\mathbf{z} = \mathbf{U}\mathbf{u}$ where $\mathbf{u} \geq \mathbf{0}$. In a similar way the dual cone \mathcal{C}^{*0} is generated by the columns of $\mathbf{W} = -\mathbf{H}'$ and note that $\mathbf{U}'\mathbf{W} = -\mathbf{I}$.

Within the euclidian metric, \mathbb{R}^k can be partitioned into 2^k convex cones as follows: let \mathcal{J} be the collection of all possible subsets of $(1, \dots, k)$, including the empty set and the whole set. For any pair $\mathbf{i}, \mathbf{j} \in \mathcal{J}$, $\mathbf{i} \cup \mathbf{j} = (1, \dots, k)$, let $(\mathbf{U}_{\mathbf{i}}, \mathbf{W}_{\mathbf{j}})$, be the matrix whose columns are, respectively, the columns of \mathbf{U} with index in \mathbf{i} and the columns of \mathbf{W} with index in \mathbf{j} ; the columns of this matrix generate the convex cone $\mathcal{C}^*(\mathbf{i})$ whose elements, when projected onto \mathcal{C}^* , belong to the face generated by the columns of $\mathbf{U}_{\mathbf{i}}$, this face is itself a convex cone of dimension equal to the cardinality $|\mathbf{i}|$ of \mathbf{i} . Thus

$$w_{i+q}(\mathcal{C}, \mathbf{V}) = w_i(\mathcal{C}^*, \mathbf{I}) = \sum_{|\mathbf{i}|=i} P[\mathbf{z} \in \mathcal{C}^*(\mathbf{i})], \quad i = 0, 1, \dots, k$$

where q is the dimension of \mathcal{L}_0 .

To compute $P[\mathbf{z} \in \mathcal{C}^*(\mathbf{i})]$ note that $\mathbf{z} \in \mathcal{C}^*(\mathbf{i})$ if and only if $\mathbf{t} = (\mathbf{U}_{\mathbf{i}}, \mathbf{W}_{\mathbf{j}})^{-1} \mathbf{z} \geq \mathbf{0}$, in other words, the linear transformation above reduces $\mathcal{C}^*(\mathbf{i})$ into the positive orthant for the multivariate normal random variable \mathbf{t} ; thus, to compute $P[\mathbf{t} \in \mathcal{R}^{k+}]$, the only quantity we need is $\text{Var}(\mathbf{t}) = \boldsymbol{\Omega}$. Let $\boldsymbol{\Psi} = \mathbf{W}'\mathbf{W}$ and $\boldsymbol{\Phi} = (\mathbf{U}'\mathbf{U})$ and note that $\boldsymbol{\Psi} = \mathbf{D}\mathbf{V}\mathbf{D}' = \boldsymbol{\Phi}^{-1}$. It can be shown that $\boldsymbol{\Omega}$ is block diagonal with elements given by $(\boldsymbol{\Phi}_{\mathbf{ii}})^{-1}$ and $(\boldsymbol{\Psi}_{\mathbf{jj}})^{-1}$, which are related by the well known formulas for the inverse of a partitioned matrix:

$$\begin{aligned} (\boldsymbol{\Phi}_{\mathbf{ii}})^{-1} &= \boldsymbol{\Psi}_{\mathbf{ii}} - \boldsymbol{\Psi}_{\mathbf{ij}}(\boldsymbol{\Psi}_{\mathbf{jj}})^{-1}\boldsymbol{\Psi}_{\mathbf{ji}} \\ (\boldsymbol{\Psi}_{\mathbf{jj}})^{-1} &= \boldsymbol{\Phi}_{\mathbf{jj}} - \boldsymbol{\Phi}_{\mathbf{ji}}(\boldsymbol{\Phi}_{\mathbf{ii}})^{-1}\boldsymbol{\Phi}_{\mathbf{ij}}. \end{aligned}$$

So, if $|\mathbf{i}| \leq |\mathbf{j}|$, it is convenient to compute $(\boldsymbol{\Phi}_{\mathbf{ii}})^{-1}$ directly and $(\boldsymbol{\Psi}_{\mathbf{jj}})^{-1}$ from the second expression above, instead, when $|\mathbf{i}| > |\mathbf{j}|$, compute $(\boldsymbol{\Psi}_{\mathbf{jj}})^{-1}$ directly and $(\boldsymbol{\Phi}_{\mathbf{ii}})^{-1}$ from the first expression above. In any case, because $\boldsymbol{\Omega}$ is block diagonal, $P[\mathbf{t} \in \mathcal{R}^{k+}]$ factorizes into the product of two lower dimensional integrals.

Because Proposition 3.6.1(3) in Silvapulle and Sen (2005) says that the weights with index j even or odd sum to 0.5, we may avoid computing the two weight which correspond to the largest number of side cones; these correspond to $(k/2 - 1, k/2)$ when k is even and to $((k-1)/2, (k+1)/2)$ when k is odd.

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